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OPTIMAL SUBPARAMETRIC FINITE ELEMENT METHOD FOR ELLIPTIC PDE OVER CIRCULAR DOMAIN

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Abstract. This paper gives us an insight into the usefulness of the proposed method in Finite Element Analysis (FEM) for solving an elliptic Partial Differential Equation (PDE) over circular domain. We are using quadratic and cubic order curved triangular elements to solve the problem of stress concentration on a circular plate, which is governed by Poisson's equation. The proposed FEM solution matches very well with the exact solution. This shows the efficiency and effectiveness of the method in various mechanical applications.

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1. Introduction

A large number of mechanical problems are governed by the partial differential equations, for example fluid flow, stress analysis, heat transfer etc. FEM helps to solve these partial differential equations by dividing the geometry into smaller elements and using numerical methods to give a solution with minimum error. The challenge for the present generation commercial software is to solve the major applications which contain irregular boundaries with less computational cost and more accuracy. The idea of curved elements in analysis of structure is given by Ergatoudis *et al* [1]. The curved boundaries are matched using the parabolic arcs; see the works of McLeod and Mitchell [2], Rathod *et al* [3, 4, 5], Nagaraja *et al* [6, 7], and Kesavulu *et al* [8]. The curved triangular elements were developed with the help of functions to resemble the curved boundaries [9]. A detailed description about the transformation is given in Rathod *et al* [3, 4, 5]. In this paper, we are demonstrating the effectiveness the FEM in an application of stress concentration over a circular plate using the quadratic and cubic curved triangular elements. The circular discs analysis finds its application in many places such as the concentration of the force in clutch plates of cars and bikes, and also beam-twisting problem in vehicles.

2. FINITE ELEMENT METHOD FOR IRREGULAR GEOMETRY

Poisson equation of elliptic type is used to describe a good number of phenomenon's in mechanical engineering. One such phenomenon is the stress concentration over circular plate. Only a quarter circle of the entire circle is taken as it is an axisymmetric problem. In case of an axisymmetric problem the symmetry takes care of the effect of force in the entire domain.

The Prandtl stress $u(x, y)$distribution of a circular plate is given [4] by

(1) $$\nabla^2 u = -2 \quad within \quad \Omega \quad (see \quad Fig.1a)$$

(2) $$u = 0 \quad on \quad C1 \quad and \quad \frac{\partial u}{\partial n} = 0 \quad on \quad C2 \quad (see \quad Fig.1a)$$

Here Ω is the region of the domain and $C1$, $C2$ represents the boundary. The exact solution of the partial differential equation is given by

(3) $$u\,(x, y) = -\,0.5\left(x^2 + y^2 - 1\right)$$

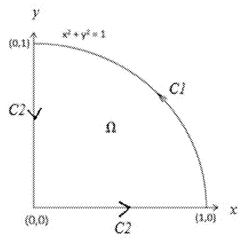

Fig. 1a. Physical representation of the domain.

The transformation of curved triangular elements for quadratic and cubic order elements with one curved side and two straight sides are given in Rathod *et al* [5]. In order to solve this type of problem, the Galerkin weighted residual method is used see Zienkiewicz *et al* [10], Reddy [11], Nagaraja *et al* [6,7] and Kesavulu *et al* [8] . The detailed procedure for applying the FEM to solve this type of problem and advantages of the proposed method are given in Nagaraja *et al* [6, 8]. On using the Galerkin weighted residual method in Eq. (1) using the quadratic and cubic order curved triangular elements, we get a system of linear algebraic equations as follows

(4) $$[K]_{NP\times NP} \times \{U\}_{NP\times 1} = \{F\}_{NP\times 1}$$

where

(5) $$K_{i,j}{=}K^{i,j}_{x,x}{+}K^{i,j}_{y,y}{=}\iint_{\Omega_e}\left(\frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial x}{+}\frac{\partial N_i}{\partial y}\frac{\partial N_j}{\partial y}\right)\text{dxdy}$$

(6)
$$K^{i,j}_{x,x}{=}\int_0^1\int_0^{1-\xi}\left(\frac{1}{J}\right)\left(\frac{\partial N_i}{\partial \xi}\frac{\partial y}{\partial n}{-}\frac{\partial N_i}{\partial \eta}\frac{\partial y}{\partial \xi}\right)\left(\frac{\partial N_j}{\partial \xi}\frac{\partial y}{\partial n}{-}\frac{\partial N_j}{\partial \eta}\frac{\partial y}{\partial \xi}\right)d\eta d\xi$$

(7)
$$K^{i,j}_{y,y}{=}\int_0^1\int_0^{1-\xi}\left(\frac{1}{J}\right)\left(-\frac{\partial N_i}{\partial \xi}\frac{\partial x}{\partial n}{+}\frac{\partial N_i}{\partial \eta}\frac{\partial x}{\partial \xi}\right)\left(-\frac{\partial N_j}{\partial \xi}\frac{\partial x}{\partial n}{+}\frac{\partial N_j}{\partial \eta}\frac{\partial x}{\partial \xi}\right)d\eta d\xi$$

$$F_i = \iint_{\Omega_e} 2N_i \; dxdy = \int_{\xi=0}^{1} \int_{\eta=0}^{1-\xi} 2JN_i\left(\xi,\eta\right) d\eta d\xi \tag{8}$$

The numerical integration is done by using Gauss quadrature rules over a standard triangle [12]. The discretization of the domain is shown in the Fig. 1b, 1c, 1d, 1e.

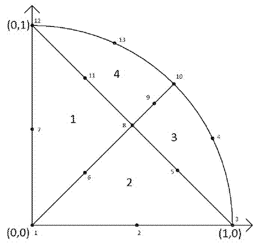

Fig. 1b. Division into 4 elements with quadratic order curved triangles.

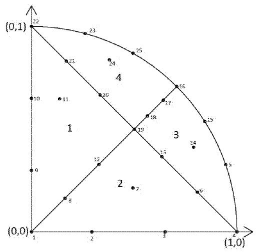

Fig. 1c. Division into 4 elements with cubic order curved triangles.

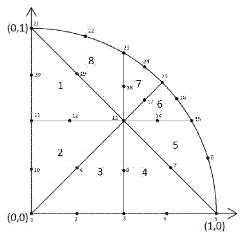

Fig. 1d. Division into 8 Elements with quadratic order triangles.

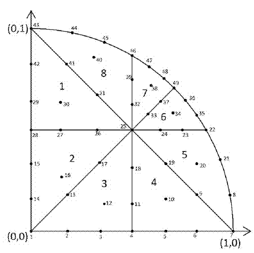

Fig. 1e. Division into 8 elements with cubic order triangles.

3. Results and Discussion

The computational experiment for solving this complex curved domain is started with a four element discretization then with eight element discretization to obtain the accuracy of two decimals. We are obtaining two decimal accuracy in using eight cubic order triangular elements. The contour plots of the stress distribution for eight cubic order triangular elements are given in Fig. 2a and the exact solution contour plot is given in Fig 2b. These two figures compare very well with each other. On comparison of values obtained for the stresses at different nodal points in the plate a very close convergence is found, which verifies the efficiency of the method used.
The current commercial software is not able to achieve two decimal accuracy with eight element discretization, since the backend program is designed in such a way to incorporate straight sided elements only. We are using straight sided elements thousands in number to achieve this accuracy in the preset software's. Hence with less number of curved triangular elements we are able to achieve the desired accuracy of two decimals. Thus this method drastically reduces the computational cost and time in finding the solution for elliptic PDE involving irregular geometry.

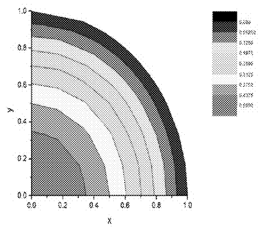

Fig. 2a. Contour plot for stress distribution with 8 element cubic order triangular elements (FEM solution).

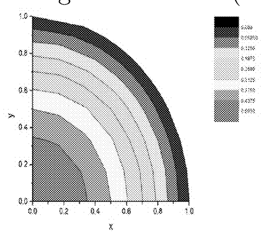

Fig. 2b. Contour plot for stress distribution using exact solution function.

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